

Non abelian $\mathcal{N} = 2$ supersymmetric Born Infeld action

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Abstract

We present a $\mathcal{N} = 2$ supersymmetric action for the Born Infeld theory in the non abelian case. We quantize the theory in $\mathcal{N} = 1$ superspace and compute divergences at one-loop. The result is discussed in the $\mathcal{N} = 4$ case.

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1 Introduction

The Born Infeld theory describes the open string tree level effective action in the approximation of slowly varying field strengths [1, 2]. It also appears in the analysis of the low-energy dynamics of a D-brane [3]. In both cases it is sufficient to consider an abelian theory in terms of a $U(1)$ gauge field. When many D-branes are put together and let coincide the gauge group becomes $U(N)$ [4]. The Born Infeld action to be studied becomes correspondingly a non abelian one. The replacement of the abelian field strength $F_{\mu\nu}$ by a non abelian tensor field is not uniquely defined. The problem has been extensively studied and discussed [5, 6, 7, 8, 9, 10]. Essentially it has to do with the ordering ambiguity in the group trace operation and in addition with the fact that derivatives acting on the field strength cannot be completely separated out since $[D_\rho, D_\sigma]F_{\mu\nu} = [F_{\rho\sigma}, F_{\mu\nu}]$.

The proposal in ref. [6] is the simplest non abelian extension of the abelian Born Infeld action. For its intrinsic elegance it is appealing by itself; in fact it exhibits also many other advantages. In particular it exactly matches the full non abelian open string effective action at least up to order α'^2 , i.e. it correctly reproduces the terms $F^2 + \alpha'^2 F^4$. It might represent a good approximation for a string dynamics where nearly covariantly constant field strengths are relevant.

The abelian bosonic Born Infeld theory admits supersymmetric $\mathcal{N} = 1$ and $\mathcal{N} = 2$ versions. A non abelian $\mathcal{N} = 1$ supersymmetrization of the theory has been proposed in ref. [11]: it has been defined in such a way that the bosonic part of the action reduces to the one in [6].

In this paper we extend the construction presented in [11, 12] to the $\mathcal{N} = 2$ non abelian Born Infeld theory. We do that in terms of $\mathcal{N} = 2$ superfields which allow an almost direct and straightforward generalization of the $\mathcal{N} = 1$ case. Indeed using the results in ref. [13], and rewriting appropriately the formulas obtained in [11] we reach our goal most easily. Then we consider the quartic interaction term, express it in terms of $\mathcal{N} = 1$ superfields and check its bosonic content. We show that it matches the F^4 terms in the bosonic non abelian action of ref. [6].

We proceed in analogy with what has been done in [14] for the abelian $\mathcal{N} = 2$ theory: we perform the quantization in $\mathcal{N} = 1$ superspace and consider $\mathcal{O}(\alpha'^4)$ one-loop corrections to the on-shell effective action. We determine the structure of the counterterm which is proportional to derivatives of the field strength. Finally, even if the complete Born Infeld action is not known for the $\mathcal{N} = 4$ case, nonetheless the $\mathcal{O}(\alpha'^4)$ one-loop result is easily computable. It can be written in a very symmetric

and elegant form. Since the corresponding contribution in the abelian theory was consistent with effective action calculations from superstring theory, it is suggestive to expect the non abelian result be in accordance with super D -brane dynamics.

We work in superspace following the notations and conventions in ref. [15] and [16]. First we briefly review the construction of the Born Infeld action in the abelian case.

The $\mathcal{N} = 1$ superfields of interest are the chiral field strengths W^α , $\bar{W}_{\dot{\alpha}}$, in terms of which the supersymmetric action can be written as [17]

$$S_{BI}^{(1)} = \int d^4x d\theta^2 W^2 + \int d^4x d\bar{\theta}^2 \bar{W}^2 + \int d^4x d\theta^2 d\bar{\theta}^2 W^\alpha W_\alpha \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} \mathcal{B}(K, \bar{K}) \quad (1)$$

where

$$\mathcal{B}(K, \bar{K}) = \left[1 - \frac{K + \bar{K}}{2} + \sqrt{1 - (K + \bar{K}) + \frac{1}{4}(K - \bar{K})^2} \right]^{-1} \quad (2)$$

and

$$K = D^2(W^\alpha W_\alpha) \quad \bar{K} = \bar{D}^2(\bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}) \quad (3)$$

It is an easy matter to check that the bosonic part of this action just reproduces the standard Born Infeld action. Indeed one can proceed as follows: first one introduces the field components of W^α defined as

$$\lambda_\alpha = W^\alpha| \quad f_{\alpha\beta} = \frac{1}{2}D_{(\alpha}W_{\beta)}| \quad D' = -\frac{i}{2}D^\alpha W_\alpha| \quad (4)$$

where $|$ indicates setting $\theta^\alpha = \bar{\theta}^{\dot{\alpha}} = 0$. In particular using the definitions

$$(\sigma_\mu)_{\alpha\dot{\alpha}} = (\mathbb{1}, \vec{\sigma}) \quad (\tilde{\sigma}_\mu)^{\dot{\alpha}\alpha} = (\mathbb{1}, -\vec{\sigma}) \quad (5)$$

and

$$(\sigma_{\mu\nu})_\alpha^\beta \equiv -\frac{1}{4}(\sigma_\mu \tilde{\sigma}_\nu - \sigma_\nu \tilde{\sigma}_\mu)_\alpha^\beta \quad (\tilde{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \equiv -\frac{1}{4}(\tilde{\sigma}_\mu \sigma_\nu - \tilde{\sigma}_\nu \sigma_\mu)^{\dot{\alpha}}_{\dot{\beta}} \quad (6)$$

the electromagnetic antisymmetric tensor can be expressed in terms of the two symmetric bispinors

$$F_{\mu\nu} = (\sigma_{\mu\nu})_{\alpha\beta} f^{\alpha\beta} - (\tilde{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}} \bar{f}^{\dot{\alpha}\dot{\beta}} \quad (7)$$

Then with the definitions

$$F^2 \equiv F^{\mu\nu} F_{\mu\nu} \quad F^4 \equiv F_{\mu\nu} F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu} \quad (8)$$

one obtains

$$\begin{aligned}
-(K + \bar{K})| &= \frac{1}{2}F^2 \equiv X \\
(K - \bar{K})^2| &= -F^4 + \frac{1}{2}(F^2)^2 \equiv -4Y^2 \\
W^2|_F + \overline{W}^2|_F &= -\frac{1}{4}F^2 \equiv I_2 \\
\frac{1}{2}(W^\alpha W_\alpha \overline{W}^{\dot{\alpha}} \overline{W}_{\dot{\alpha}})|_D &= \frac{1}{8}(F^4 - \frac{1}{4}(F^2)^2) \equiv I_4
\end{aligned} \tag{9}$$

Finally in terms of the above invariants the bosonic Born Infeld lagrangian takes the form

$$\begin{aligned}
L_{BI} &= \left(1 - \sqrt{-\det_4(\eta_{\mu\nu} + F_{\mu\nu})}\right) \\
&= I_2 + 2I_4 \left[1 + \frac{X}{2} + \sqrt{1 + X - Y^2}\right]^{-1}
\end{aligned} \tag{10}$$

The expression in (10), with the identifications in (9), clearly coincides with the bosonic part of the $\mathcal{N} = 1$ supersymmetric action in (1).

The same approach allows one to obtain the $\mathcal{N} = 2$ supersymmetric abelian version of the action [13]¹. The $\mathcal{N} = 2$ superfields of interest are the chiral field strengths \mathcal{W} and $\overline{\mathcal{W}}$ which satisfy the constraints

$$D_a^\alpha D_{b\alpha} \mathcal{W} = C_{ac} C_{bd} \overline{D}^{d\dot{\alpha}} \overline{D}_{\dot{\alpha}}^c \overline{\mathcal{W}} \tag{11}$$

where C_{ab} is the Levi-Civita antisymmetric tensor. The $\mathcal{N} = 1$ components are defined as

$$\phi \equiv \mathcal{W}| \quad W_\alpha \equiv -D_{2\alpha} \mathcal{W}| \tag{12}$$

and everything is evaluated at $\theta_2^\alpha = \bar{\theta}_2^{\dot{\alpha}} = 0$. In complete correspondence with the equations in (1), (2) and (3) one has

$$S_{BI}^{(2)} = \frac{1}{2} \int d^4x d\theta^4 \mathcal{W}^2 + \frac{1}{2} \int d^4x d\bar{\theta}^4 \overline{\mathcal{W}}^2 + \int d^4x d\theta^4 d\bar{\theta}^4 \mathcal{W} \mathcal{W} \overline{\mathcal{W}} \overline{\mathcal{W}} \mathcal{Y}(K, \bar{K}) \tag{13}$$

where

$$\mathcal{Y}(\mathcal{K}, \bar{\mathcal{K}}) = \left[1 - \frac{\mathcal{K} + \bar{\mathcal{K}}}{2} + \sqrt{1 - (\mathcal{K} + \bar{\mathcal{K}}) + \frac{1}{4}(\mathcal{K} - \bar{\mathcal{K}})^2}\right]^{-1} \tag{14}$$

¹ Note however that beyond quartic interactions the form of the action is not unique, see for example [18]

and

$$\mathcal{K} = D^4 \mathcal{W}^2 \quad \bar{\mathcal{K}} = \bar{D}^4 \bar{\mathcal{W}}^2 \quad (15)$$

Now we turn to the construction for the non abelian case. The various actions which appear in (10), (1), (13) contain all order interaction terms obtained by the power series expansion of the square root. In order to promote these theories from abelian to non abelian ones it is sufficient to introduce gauge covariant derivatives, treat $F^{\mu\nu}$, W^α , \mathcal{W} as matrices, expand the square root as before and take the trace of the various terms. In order to overcome the ordering ambiguity, in [6] it has been suggested to introduce a symmetrized trace defined for any set of matrices A_1, A_2, \dots, A_n as

$$\text{STr}(A_1, A_2, \dots, A_n) = \frac{1}{n!} \sum_{perm.} \text{Tr}(A_{\sigma_1}, A_{\sigma_2}, \dots, A_{\sigma_n}) \quad (16)$$

For the bosonic action in (10) we then obtain the non abelian generalization

$$\begin{aligned} L_{BI} &= \text{STr} \left(1 - \sqrt{-\det_4(\eta_{\mu\nu} + F_{\mu\nu})} \right) \\ &= \sum_{n=0}^{\infty} q_n \text{STr}[(X - Y^2)^{n+1}] = \sum_{n=1}^{\infty} q_{n-1} \sum_{j=0}^n \binom{n}{j} \text{STr}[X^j (-Y^2)^{n-j}] \end{aligned} \quad (17)$$

with

$$q_0 = -\frac{1}{2} \quad q_n = \frac{(-1)^{n+1}}{4^n} \frac{(2n-1)!}{(n+1)!(n-1)!} \quad (18)$$

Correspondingly in the $\mathcal{N} = 1$ case, from (1) we write

$$\begin{aligned} S_{BI}^{(1)} &= \int d^4x \, d\theta^2 \, \text{Tr} W^2 + \int d^4x \, d\bar{\theta}^2 \, \text{Tr} \bar{W}^2 \\ &\quad + \sum_{n,m} \frac{C_{n,m}}{2} \int d^4x \, d\theta^2 d\bar{\theta}^2 \, \text{STr}(W^\alpha, W_\alpha, \bar{W}^{\dot{\alpha}}, \bar{W}_{\dot{\alpha}}, \hat{X}^n, \hat{Y}^m) \end{aligned} \quad (19)$$

where

$$\begin{aligned} \hat{X} &= -(K + \bar{K}) & \hat{X}| &= X \\ \hat{Y} &= -\frac{i}{2}(K - \bar{K}) & \hat{Y}| &= Y \end{aligned} \quad (20)$$

The lowest order interaction is given by

$$L_4 = \frac{1}{2} \text{STr}(W^\alpha, W_\alpha, \bar{W}^{\dot{\alpha}}, \bar{W}_{\dot{\alpha}}) = \frac{1}{3} \text{Tr}(W^\alpha W_\alpha \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} - \frac{1}{2} W^\alpha \bar{W}^{\dot{\alpha}} W_\alpha \bar{W}_{\dot{\alpha}}) \quad (21)$$

whose bosonic expression exactly reproduces the non abelian bosonic terms proposed in [6]. Indeed one obtains

$$\begin{aligned}
L_{4 \text{ bos}} &= \frac{1}{3} \text{Tr}(f^{\alpha\beta} f_{\alpha\beta} \bar{f}^{\dot{\alpha}\dot{\beta}} \bar{f}_{\dot{\alpha}\dot{\beta}} - \frac{1}{2} f^{\alpha\beta} \bar{f}^{\dot{\alpha}\dot{\beta}} f_{\alpha\beta} \bar{f}_{\dot{\alpha}\dot{\beta}}) \\
&= \frac{1}{12} \text{Tr}(F^{\mu\nu} F_{\nu\sigma} F_{\mu\rho} F^{\rho\sigma} + \frac{1}{2} F^{\mu\nu} F_{\nu\sigma} F^{\rho\sigma} F_{\mu\rho} \\
&\quad - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} - \frac{1}{8} F^{\mu\nu} F^{\rho\sigma} F_{\mu\nu} F_{\rho\sigma}) \\
&= \frac{1}{8} \text{STr}(F^4 - \frac{1}{4} (F^2)^2)
\end{aligned} \tag{22}$$

Now one has to determine the higher order terms which appear in (19). First of all one has to consider the symmetrized trace of powers of n, m matrices $\widehat{X}^n, \widehat{Y}^m$. Moreover for the superfields K e \bar{K} contained in \widehat{X} e \widehat{Y} one has to use the decomposition

$$\nabla^2(W^\alpha W_\alpha) = (\nabla^2 W^\alpha) W_\alpha + W^\alpha (\nabla^2 W_\alpha) - \nabla^\alpha W^\beta \nabla_\alpha W_\beta \tag{23}$$

and remember that the STr operation permutes each W_α factor.

Then the coefficients $C_{n,m}$ need to be computed. Even if in ref. [11], [12] they have been evaluated for the $\mathcal{N} = 1$ theory, since we are using a different notation we give here our derivation that immediately extends to the $\mathcal{N} = 2$ case. The problem is to compare the following two series

$$\begin{aligned}
L_{BI} &= \sum_{n=1}^{\infty} q_{n-1} \sum_{j=0}^n \binom{n}{j} X^{n-j} (-Y^2)^j \\
&= \sum_{n,m} (q_1 X^2 - q_0 Y^2) C_{n,m} X^n Y^m
\end{aligned} \tag{24}$$

We immediately have $C_{n,2m+1} = 0$. Then since $q_0 = -\frac{1}{2}$, $q_1 = \frac{1}{8}$ one can rewrite

$$\begin{aligned}
L_{BI} &= \sum_{n,m} (q_1 C_{n,2m} X^{n+2} (Y^2)^m - q_0 C_{n,2m} X^n (Y^2)^{m+1}) \\
&= \sum_{n,m} \left(\frac{C_{n-m-2,2m}}{8} + \frac{C_{n-m,2m-2}}{2} \right) X^{n-m} (Y^2)^m
\end{aligned} \tag{25}$$

From (24) one obtains the recursive relation

$$\frac{C_{n-j-2,2j}}{8} + \frac{C_{n-j,2j-2}}{2} = (-1)^j q_{n-1} \binom{n}{j} \tag{26}$$

which gives

$$C_{n-2j,2j} = 8(-1)^j q_{n-j+1} \binom{n-j+2}{j} - 4C_{n-2j+2,2j-2} \tag{27}$$

The solution of the above equation determines the wanted coefficients

$$\begin{aligned}
C_{n-2j,2j} &= 8(-1)^j q_{n-j+1} \binom{n-j+2}{j} - 8 \cdot 4(-1)^{j-1} q_{n-j+2} \binom{n-j+3}{j-1} + \dots \\
&= 8(-1)^j \sum_{k=0}^j 4^{j-k} \binom{n+2-k}{k} q_{n+1-k}
\end{aligned} \tag{28}$$

Now, if for the $\mathcal{N} = 2$ case, we define in complete analogy with what we have done so far

$$\widehat{\mathcal{X}} = -(\mathcal{K} + \bar{\mathcal{K}}) \quad \widehat{\mathcal{Y}} = -\frac{i}{2}(\mathcal{K} - \bar{\mathcal{K}}) \tag{29}$$

we can write the non abelian generalization of the supersymmetric $\mathcal{N} = 2$ action in the form

$$\begin{aligned}
S_{BI}^{(2)} &= \frac{1}{2} \int d^4x \, d\theta^4 \, \text{Tr}(\mathcal{W}^2) + \frac{1}{2} \int d^4x \, d\bar{\theta}^4 \, \text{Tr}(\bar{\mathcal{W}}^2) \\
&\quad + \sum_{n,m} \frac{C_{n,m}}{2} \int d^4x \, d\theta^4 \, d\bar{\theta}^4 \, \text{STr}(\mathcal{W}, \mathcal{W}, \bar{\mathcal{W}}, \bar{\mathcal{W}}, (\widehat{\mathcal{X}})^n, (\widehat{\mathcal{Y}})^m)
\end{aligned} \tag{30}$$

with the $C_{n,m}$ coefficients given in (28).

Although the action in (30) is explicit, the computation of the interaction terms becomes quite cumbersome as soon as one goes to higher order in the superfield expansion.

Here we concentrate on the quartic terms and express them in terms of $\mathcal{N} = 1$ superfields. Then we will quantize the action in $\mathcal{N} = 1$ superspace and compute perturbative corrections to the $\mathcal{O}(\alpha'^4)$ in the spirit of [14]. The quartic interaction is given by

$$L_4 = \frac{1}{3} \text{Tr}[\mathcal{W}\mathcal{W}\bar{\mathcal{W}}\bar{\mathcal{W}}] + \frac{1}{2} \mathcal{W}\bar{\mathcal{W}}\mathcal{W}\bar{\mathcal{W}} \tag{31}$$

The reduction to $\mathcal{N} = 1$ superspace is obtained by projection performed now in terms of gauge covariant derivatives [16]. The commutator algebra is given by

$$\begin{aligned}
\{\nabla_{a\alpha}, \nabla_{b\beta}\} &= iC_{ab}C_{\alpha\beta}\bar{\mathcal{W}} \quad \{\bar{\nabla}_{\dot{\alpha}}^a, \bar{\nabla}_{\dot{\beta}}^b\} = iC^{ab}C_{\dot{\alpha}\dot{\beta}}\mathcal{W} \\
\{\nabla_{a\alpha}, \bar{\nabla}_{\dot{\beta}}^b\} &= i\delta_a^b \nabla_{\alpha\dot{\beta}}
\end{aligned} \tag{32}$$

where the superfields \mathcal{W} and $\bar{\mathcal{W}}$ satisfy the covariant constraints

$$\nabla_a^\alpha \nabla_{b\alpha} \mathcal{W} = C_{ac} C_{bd} \bar{\nabla}^{d\dot{\alpha}} \bar{\nabla}_{\dot{\alpha}}^c \bar{\mathcal{W}} \tag{33}$$

The $\mathcal{N} = 1$ projections are

$$\phi \equiv \mathcal{W}| \quad W_\alpha \equiv -\nabla_{2\alpha} \mathcal{W}| \quad (\nabla_2)^2 \mathcal{W}| = (\bar{\nabla}^1)^2 \bar{\mathcal{W}}| = (\bar{\nabla})^2 \bar{\phi} \tag{34}$$

It is rather straightforward to obtain

$$\begin{aligned}
& \text{Tr}(\nabla_2)^2(\bar{\nabla}_2)^2(\mathcal{W}\mathcal{W}\bar{\mathcal{W}}\bar{\mathcal{W}})| = \\
& \text{Tr}\{W^\alpha W_\alpha \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} - iW^\alpha \phi \nabla_{\alpha\dot{\alpha}} \bar{\phi} \bar{W}^{\dot{\alpha}} - iW^\alpha \bar{\phi} \bar{W}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} \phi - i\phi W^\alpha \nabla_{\alpha\dot{\alpha}} \bar{\phi} \bar{W}^{\dot{\alpha}} \\
& - i\phi W^\alpha \bar{W}^{\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} \phi + \bar{\nabla}^2 \bar{\phi} \bar{\phi} \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} + \phi \bar{\nabla}^2 \bar{\phi} \bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}} + W^\alpha W_\alpha \nabla^2 \phi \bar{\phi} \\
& + W^\alpha W_\alpha \bar{\phi} \nabla^2 \phi - iW^\alpha \phi \bar{\phi} \nabla_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}} - i\phi W^\alpha \bar{\phi} \nabla_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}} - iW^\alpha \phi \nabla_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}} \bar{\phi} \\
& - i\phi W^\alpha \nabla_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}} \bar{\phi} + \bar{\nabla}^2 \bar{\phi} \phi \nabla^2 \phi \bar{\phi} + \phi \bar{\nabla}^2 \bar{\phi} \nabla^2 \phi \bar{\phi} + \bar{\nabla}^2 \bar{\phi} \phi \phi \nabla^2 \phi \\
& + \phi \bar{\nabla}^2 \bar{\phi} \phi \nabla^2 \phi + \phi \phi \square \bar{\phi} \bar{\phi} + \phi \phi \phi \square \bar{\phi} + \phi \phi \nabla^{\alpha\dot{\alpha}} \bar{\phi} \nabla_{\alpha\dot{\alpha}} \bar{\phi} \\
& - \frac{3}{2} i\phi \phi \bar{\nabla}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \bar{\phi} \bar{\phi} + \frac{3}{2} i\phi \phi \bar{\phi} \bar{\nabla}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} + i\phi \phi \bar{\nabla}^{\dot{\alpha}} \bar{\phi} \bar{W}_{\dot{\alpha}} \bar{\phi} + i\phi \phi \bar{\phi} \bar{W}_{\dot{\alpha}} \bar{\nabla}^{\dot{\alpha}} \bar{\phi} \\
& - i\phi \phi \bar{\nabla}_{\dot{\alpha}} \bar{\phi} \bar{W}^{\dot{\alpha}} - i\phi \phi \bar{W}^{\dot{\alpha}} \bar{\phi} \bar{\nabla}_{\dot{\alpha}} \bar{\phi} + iW^\alpha \nabla_\alpha \phi \phi \bar{\phi} \bar{\phi} + i\nabla_\alpha \phi W^\alpha \bar{\phi} \bar{\phi} \bar{\phi} \\
& - i\phi W^\alpha \nabla_\alpha \phi \bar{\phi} \bar{\phi} + iW^\alpha \phi \bar{\phi} \bar{\phi} \nabla_\alpha \phi + \phi \phi [\bar{\phi}, [\bar{\phi}, \phi]] \bar{\phi} + \phi \phi \phi [\bar{\phi}, [\bar{\phi}, \phi]] \} \quad (35)
\end{aligned}$$

The last set of terms is zero in the abelian case (cf.[14].) In the same way one has

$$\begin{aligned}
& \text{Tr}\{\frac{1}{2}(\nabla_2)^2(\bar{\nabla}_2)^2(\mathcal{W}\bar{\mathcal{W}}\mathcal{W}\bar{\mathcal{W}})|\} = \\
& \text{Tr}\{-\frac{1}{2}W^\alpha \bar{W}^{\dot{\alpha}} W_\alpha \bar{W}_{\dot{\alpha}} - iW^\alpha \bar{W}^{\dot{\alpha}} \phi \nabla_{\alpha\dot{\alpha}} \bar{\phi} - iW^\alpha \nabla_{\alpha\dot{\alpha}} \bar{\phi} \phi \bar{W}^{\dot{\alpha}} \\
& + W^\alpha \nabla^2 \phi W_\alpha \bar{\phi} + \bar{\nabla}^2 \bar{\phi} \bar{W}^{\dot{\alpha}} \phi \bar{W}_{\dot{\alpha}} - iW^\alpha \nabla_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}} \phi \bar{\phi} - iW^\alpha \bar{\phi} \phi \nabla_{\alpha\dot{\alpha}} \bar{W}^{\dot{\alpha}} \\
& \bar{\nabla}^2 \bar{\phi} \nabla^2 \phi \phi \bar{\phi} + \phi \nabla^2 \phi \bar{\nabla}^2 \bar{\phi} \bar{\phi} + \frac{1}{2} \phi \nabla_{\alpha\dot{\alpha}} \bar{\phi} \phi \nabla^{\alpha\dot{\alpha}} \bar{\phi} + \frac{1}{2} \phi \nabla^{\alpha\dot{\alpha}} \nabla_{\alpha\dot{\alpha}} \bar{\phi} \phi \bar{\phi} \\
& + iW^\alpha \bar{\phi} \nabla_\alpha \phi \phi \bar{\phi} - iW^\alpha \nabla_\alpha \phi \bar{\phi} \phi \bar{\phi} + i\phi \bar{\phi} \nabla^\alpha \phi W_\alpha \bar{\phi} - i\phi \nabla^\alpha \phi \bar{\phi} W_\alpha \bar{\phi} \\
& + \frac{i}{2} \phi \bar{\phi} \phi \bar{\phi} \nabla^\alpha W_\alpha - \frac{i}{2} \phi \bar{\phi} \phi \nabla^\alpha W_\alpha \bar{\phi} + i\phi \bar{\phi} \phi \bar{\nabla}^{\dot{\alpha}} \bar{\phi} \bar{W}_{\dot{\alpha}} + i\phi \bar{\phi} \phi \bar{W}_{\dot{\alpha}} \bar{\nabla}^{\dot{\alpha}} \bar{\phi} \\
& - i\phi \phi \bar{\nabla}^{\dot{\alpha}} \bar{\phi} \bar{W}_{\dot{\alpha}} + i\phi \bar{\nabla}^{\dot{\alpha}} \bar{\phi} \bar{\phi} \phi \bar{W}_{\dot{\alpha}} + \phi(\bar{\phi} \bar{\phi} \phi + \phi \bar{\phi} \bar{\phi} - 2\bar{\phi} \phi \bar{\phi}) \phi \bar{\phi} \} \quad (36)
\end{aligned}$$

In the abelian case the last set of terms vanishes (cf.[14]). One can check that the quartic terms which contain only the W^α superfields reduce to the ones in (21).

In the last part of the paper we want to perform for the non abelian theory the same one-loop calculation performed in ref. [14], i.e. we compute the $\mathcal{O}(\alpha^4)$ on-shell divergent contributions to the effective action with four external vector field-strengths. The superspace D -algebra manipulations and the computation of the momentum integrals for the three different types of diagrams relevant for this calculations are exactly the same as in the abelian case. We do not repeat here

those steps. For that part of the calculation we simply make reference to the above mentioned paper and directly use the results obtained therein. Thus we concentrate on the non abelian group structure that arises in this new case.

The quartic vertices that enter the three graphs, G_1, G_2, G_3 as in [14], arise from the L_4 lagrangian in (31) where we write $\mathcal{W} = \mathcal{W}^a T^a$ being T^a the matrices of the gauge group. The colour structure associated to a vertex from (31) is then given by

$$\text{Tr}(T^a T^b T^c T^d + \frac{1}{2} T^a T^c T^b T^d) \quad (37)$$

It is rather straightforward to examine the contributions for the three different diagrams. They all have a bubble type topology with two external vector lines at each vertex. The interaction vertices are written explicitly in (35) and (36). G_1 and G_2 have both internal quantum vector lines, while G_3 contains quantum chiral fields. One has to use the fact that the propagators are diagonal in the colour indices and to take correctly into account all the various possibilities for the contractions of the internal quantum lines. One finds that all the three diagrams give rise to a colour factor of the form

$$R(a, b; i, j) = (3!)^2 \text{STr}(T^a T^b T^c T^d) \text{STr}(T^c T^d T^i T^j) \quad (38)$$

where a, b, i, j are the colour indices on the external fields at vertex 1 and vertex 2 respectively.

We list the answers obtained at this stage for the three graphs separately, using a self explanatory notation

$$\begin{aligned} \Gamma_{G_1}^{(4)}[W, \overline{W}] &= -\frac{1}{3^2} \frac{1}{2} \underbrace{W^{\alpha a}(-p_1) W_{\alpha}^b(-p_2)}_1 \underbrace{\overline{W}^{\dot{\alpha} i}(-p_3) \overline{W}_{\dot{\alpha}}^j(-p_4)}_2 \\ &\quad \langle \overline{W}^{\dot{\beta}}(1) W^{\beta}(2) \rangle \langle \overline{W}_{\dot{\beta}}(1) W_{\beta}(2) \rangle \cdot R(a, b; i, j) \end{aligned} \quad (39)$$

$$\begin{aligned} \Gamma_{G_2}^{(4)}[W, \overline{W}] &= \frac{1}{3^2} \underbrace{W_{\alpha}^a(-p_1) \overline{W}_{\dot{\alpha}}^i(-p_3)}_1 \underbrace{W_{\beta}^b(-p_2) \overline{W}_{\dot{\beta}}^j(-p_4)}_2 \\ &\quad \langle W^{\alpha}(1) \overline{W}^{\dot{\beta}}(2) \rangle \langle \overline{W}^{\dot{\alpha}}(1) W^{\beta}(2) \rangle \cdot R(a, i; b, j) \\ &\quad + (p_3 \Leftrightarrow p_4; i \Leftrightarrow j) \end{aligned} \quad (40)$$

$$\begin{aligned} \Gamma_{G_3}^{(4)}[W, \overline{W}] &= \frac{1}{3^2} \underbrace{W^{\alpha a}(-p_1) \overline{W}^{\dot{\alpha} i}(-p_3)}_1 \underbrace{W^{\beta b}(-p_2) \overline{W}^{\dot{\beta} j}(-p_4)}_2 \cdot \\ &\quad \langle \phi(1) i \partial_{\beta \dot{\beta}} \bar{\phi}(2) \rangle \langle i \partial_{\alpha \dot{\alpha}} \bar{\phi}(1) \phi(2) \rangle \cdot R(a, i; b, j) \\ &\quad + (p_3 \Leftrightarrow p_4; i \Leftrightarrow j) \end{aligned} \quad (41)$$

Then we perform the D -algebra and extract the divergent part from the momentum integrals exactly as in [14] and obtain:

For the diagram G_1

$$\Gamma_{G_1}^{(4) \text{ div.}} [W, \overline{W}] = \frac{\alpha^4}{18} \frac{1}{\epsilon} \frac{1}{(4\pi)^2} \int d^4 p_1 \dots d^4 p_4 \delta(\Sigma p_i) d^2 \theta d^2 \overline{\theta} R(a, b; i, j) s^2 \left[W^{\alpha a}(-p_1) W_{\alpha}^b(-p_2) \overline{W}^{\dot{\alpha} i}(-p_3) \overline{W}_{\dot{\alpha}}^j(-p_4) \right] \quad (42)$$

with $s = (p_1 + p_2)^2$.

For the diagram G_2

$$\Gamma_{G_2}^{(4) \text{ div}} [W, \overline{W}] = \frac{\alpha^4}{18} \frac{1}{2} \frac{1}{\epsilon} \frac{1}{(4\pi)^2} \int d^4 p_1 \dots d^4 p_4 \delta(\Sigma p_i) d^2 \theta d^2 \overline{\theta} R(a, i; b, j) t^2 \left[W^{\alpha a}(-p_1) W_{\alpha}^b(-p_2) \overline{W}^{\dot{\alpha} i}(-p_3) \overline{W}_{\dot{\alpha}}^j(-p_4) \right] + (t^2 \Leftrightarrow u^2; i \Leftrightarrow j) \quad (43)$$

where $t = (p_1 + p_3)^2$ and $u = (p_1 + p_4)^2$.

For the diagram G_3

$$\Gamma_{G_3}^{(4) \text{ div}} [W, \overline{W}] = \frac{\alpha^4}{18} \frac{1}{6} \frac{1}{\epsilon} \frac{1}{(4\pi)^2} \int d^4 p_1 \dots d^4 p_4 \delta(\Sigma p_i) d^2 \theta d^2 \overline{\theta} R(a, i; b, j) t^2 \left[W^{\alpha a}(-p_1) W_{\alpha}^b(-p_2) \overline{W}^{\dot{\alpha} i}(-p_3) \overline{W}_{\dot{\alpha}}^j(-p_4) \right] + (t^2 \Leftrightarrow u^2; i \Leftrightarrow j) \quad (44)$$

Finally the complete result is

$$\Gamma_{\text{div}}^{(4)} [W, \overline{W}] = \frac{\alpha^4}{18} \frac{1}{\epsilon} \frac{1}{(4\pi)^2} \int d^4 p_1 \dots d^4 p_4 \delta(\Sigma p_i) d^2 \theta d^2 \overline{\theta} \left[W^{\alpha a}(-p_1) W_{\alpha}^b(-p_2) \overline{W}^{\dot{\alpha} i}(-p_3) \overline{W}_{\dot{\alpha}}^j(-p_4) \right] \left[s^2 R(a, b; i, j) + \frac{2}{3} t^2 R(a, i; b, j) + \frac{2}{3} u^2 R(a, j; b, i) \right] \quad (45)$$

The above expression can be rewritten in configuration space and made more explicit e.g. computing the $R(a, b; i, j)$ factors for a specific gauge group. These manipulations amount to rather simple exercises.

Now we concentrate on the extension of the above result to the $\mathcal{N} = 4$ case.

As already mentioned the complete $\mathcal{N} = 4$ supersymmetric Born Infeld action is still unknown. What is available are the quartic vertices of the abelian action [9] written in terms of $\mathcal{N} = 1$ superfields. As compared to the $\mathcal{N} = 2$ case there

appear three chiral superfields instead of one. For the one-loop calculation we have reported the only change will be in the diagram G_3 which contains chiral superfields propagating in the loop. In order to obtain the corresponding $\mathcal{N} = 4$ contribution it is sufficient to multiply by three the result in (44), while the contributions from the G_1 and G_2 graphs do not change. Thus summing (42), (43) and three times (44) we obtain

$$\begin{aligned} \Gamma_{div}^{(4)}[W, \overline{W}] &= \frac{\alpha^4}{18} \frac{1}{\epsilon} \frac{1}{(4\pi)^2} \int d^4 p_1 \dots d^4 p_4 \delta(\Sigma p_i) d^2 \theta d^2 \overline{\theta} \\ &\quad \left[W^{\alpha a}(-p_1) W_{\alpha}^b(-p_2) \overline{W}^{\dot{\alpha} i}(-p_3) \overline{W}_{\dot{\alpha}}^j(-p_4) \right] \\ &\quad \left[s^2 R(a, b; i, j) + t^2 R(a, i; b, j) + u^2 R(a, j; b, i) \right] \end{aligned} \quad (46)$$

As compared to the $\mathcal{N} = 2$ result we have now the factor

$$\left[s^2 R(a, b; i, j) + t^2 R(a, i; b, j) + u^2 R(a, j; b, i) \right] \quad (47)$$

which contains more symmetries than before, e.g. $b \leftrightarrow i$ $p_2 \leftrightarrow p_3$.

It is quite interesting to extract from the result in (46) its bosonic component content. After some not so simple algebra one obtains

$$\begin{aligned} \Gamma_{div}^{(4)}[F] &= \frac{\alpha^4}{18} \frac{1}{\epsilon} \frac{1}{(4\pi)^2} \int d^4 p_1 \dots d^4 p_4 \delta(\Sigma p_i) \\ &\quad \frac{1}{96} (t_8)^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} F_{\mu_1 \nu_1}^a(-p_1) F_{\mu_2 \nu_2}^b(-p_2) F_{\mu_3 \nu_3}^i(-p_3) F_{\mu_4 \nu_4}^j(-p_4) \\ &\quad \left[s^2 R(a, b; i, j) + t^2 R(a, i; b, j) + u^2 R(a, j; b, i) \right] \end{aligned} \quad (48)$$

where we have introduced the tensor [19]

$$\begin{aligned} t_8^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} &= -\frac{1}{2} \epsilon^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} \\ &\quad -\frac{1}{2} [(\delta^{\mu_1 \mu_2} \delta^{\nu_1 \nu_2} - \delta^{\mu_1 \nu_2} \delta^{\nu_1 \mu_2})(\delta^{\mu_3 \mu_4} \delta^{\nu_3 \nu_4} - \delta^{\mu_3 \nu_4} \delta^{\nu_3 \mu_4}) + \\ &\quad (\delta^{\mu_2 \mu_3} \delta^{\nu_2 \nu_3} - \delta^{\mu_2 \nu_3} \delta^{\nu_2 \mu_3})(\delta^{\mu_4 \mu_1} \delta^{\nu_4 \nu_1} - \delta^{\mu_4 \nu_1} \delta^{\nu_4 \mu_1}) + \\ &\quad (\delta^{\mu_1 \mu_3} \delta^{\nu_1 \nu_3} - \delta^{\mu_1 \nu_3} \delta^{\nu_1 \mu_3})(\delta^{\mu_2 \mu_4} \delta^{\nu_2 \nu_4} - \delta^{\mu_2 \nu_4} \delta^{\nu_2 \mu_4})] + \\ &\quad +\frac{1}{2} [\delta^{\nu_1 \mu_2} \delta^{\nu_2 \mu_3} \delta^{\nu_3 \mu_4} \delta^{\nu_4 \mu_1} + \delta^{\nu_1 \mu_3} \delta^{\nu_3 \mu_2} \delta^{\nu_2 \mu_4} \delta^{\nu_4 \mu_1} + \delta^{\nu_1 \mu_3} \delta^{\nu_3 \mu_4} \delta^{\nu_4 \mu_2} \delta^{\nu_2 \mu_1} + \\ &\quad +45 \text{ antisymmetrization}] \end{aligned} \quad (49)$$

In order to streamline the notation we define $F_{\mu\nu}^a(-p_1) \equiv F_{\mu\nu}^a$ and similarly $(b, p_2 \rightarrow b)$, $(i, p_3 \rightarrow i)$, $(j, p_4 \rightarrow j)$. Moreover for a cyclic contraction we use the notation

$F^{abij} \equiv (F^a)^{\mu\nu}(F^b)_{\nu\rho}(F^i)^{\rho\sigma}(F^j)_{\sigma\mu}$. In this way we can write

$$(t_8)^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} F_{\mu_1\nu_1}^a F_{\mu_2\nu_2}^b F_{\mu_3\nu_3}^i F_{\mu_4\nu_4}^j = 8 \left[F^{abij} + F^{aijb} + F^{ajbi} - \frac{1}{4}(F^a \cdot F^b)(F^i \cdot F^j) - \frac{1}{4}(F^a \cdot F^i)(F^j \cdot F^b) - \frac{1}{4}(F^a \cdot F^j)(F^b \cdot F^i) \right] \quad (50)$$

which in the abelian case reduces to $4!(F^4 - \frac{1}{4}(F^2)^2)$.

We can also express our result in configuration space; for example

$$\begin{aligned} s^2 R(a, b; i, j) (t_8)^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4} F_{\mu_1\nu_1}^a F_{\mu_2\nu_2}^b F_{\mu_3\nu_3}^i F_{\mu_4\nu_4}^j &\Longrightarrow \\ \Longrightarrow 32 R(a, b; i, j) [2(\partial^\mu \partial^\nu F^a)(\partial_\mu \partial_\nu F^b) F^i F^j + (\partial^\mu \partial^\nu F^a) F^i (\partial_\mu \partial_\nu F^b) F^j \\ - \frac{1}{4}(\partial^\mu \partial^\nu F^a \cdot \partial_\mu \partial_\nu F^b)(F^i \cdot F^j) - \frac{1}{2}(\partial^\mu \partial^\nu F^a \cdot F^i)(\partial_\mu \partial_\nu F^b \cdot F^j)] \end{aligned} \quad (51)$$

Then it becomes apparent that the three contributions in (48) proportional to s^2 , t^2 and u^2 are equal, so that in configuration space the final result can be written in a rather simple form

$$\begin{aligned} \Gamma_{div}^{(4)}[F] &= \frac{\alpha^4}{18} \frac{1}{\epsilon} \frac{1}{(4\pi)^2} \int d^4x R(a, b; i, j) \\ &\quad [2(\partial^\mu \partial^\nu F^a)(\partial_\mu \partial_\nu F^b) F^i F^j + (\partial^\mu \partial^\nu F^a) F^i (\partial_\mu \partial_\nu F^b) F^j \\ &\quad - \frac{1}{4}(\partial^\mu \partial^\nu F^a \cdot \partial_\mu \partial_\nu F^b)(F^i \cdot F^j) - \frac{1}{2}(\partial^\mu \partial^\nu F^a \cdot F^i)(\partial_\mu \partial_\nu F^b \cdot F^j)] \end{aligned} \quad (52)$$

The above result, when restricted to abelian fields, coincides with corresponding results in [20], [14] and it is consistent with scattering amplitude calculations in type IIB string theory on the $D3$ -brane [21] and in type I open string theory [22, 1, 23]. It would be interesting to confront the non abelian structure obtained in (52) with corresponding calculations for scattering of strings on N coinciding D -branes along the lines of refs. [21, 24].

After completion of this work the paper in ref. [25] has appeared; it contains a derivation of the non abelian $\mathcal{N} = 2$ Born Infeld action from partial breaking of $\mathcal{N} = 4$ supersymmetry.

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References

- [1] D.J. Gross, E. Witten, Nucl. Phys. **B277** (1986) 1
- [2] A.A. Tseytlin, Nucl. Phys. **B276** (1986) 391; Nucl. Phys. **B291** (1987) 876 (E)
- [3] R.G. Leigh, Mod. Phys. Lett. **A4** (1989) 2767
- [4] E. Witten, Nucl. Phys. **B460** (1996) 335, hep-th/9510135
- [5] A. Abouelsaood, C.G. Callan, C.R. Nappi, S.A. Yost, Nucl. Phys. **B280** (1987) 135
- [6] A.A. Tseytlin, Nucl. Phys. **B501** (1997) 41, hep-th/9701125
- [7] A. Hashimoto, W. Taylor IV, Nucl. Phys. **B503** (1997) 193, hep-th/9703217
- [8] D. Brecher, M.J. Perry, Nucl. Phys. **B527** (1998) 121, hep-th/9801127
- [9] A.A.Tseytlin, "Born-Infeld action, supersymmetry and string theory," hep-th/9908105
- [10] F. Denef, A. Sevrin, J. Troost, "Non-abelian Born-Infeld versus string theory", hep-th/0002180.
- [11] S. Gonorazky, F.A. Schaposnik, G. Silva, Phys. Lett. **B449** (1999) 187, hep-th/9812094
- [12] S. Gonorazky, C. Nunez, F.A. Schaposnik, G. Silva, Nucl. Phys. **B531** (1998) 168, hep-th/9805054
- [13] S.V. Ketov, Mod. Phys. Lett. **A14** (1999) 501, hep-th/9809121
- [14] A. De Giovanni, A. Santambrogio, D.Zanon, Phys. Lett. **B472** (2000) 94, hep-th/9907214
- [15] S.J. Gates, M.T. Grisaru, M. Rocek, W. Siegel, "Superspace" (Benjamin-Cummings, Reading, MA, 1983)
- [16] B. de Wit, M.T. Grisaru, M. Rocek, Phys. Lett. **B374** (1996) 297, hep-th/9601115

- [17] S. Cecotti, S. Ferrara, Phys. Lett. **B187** (1987) 335
- [18] S.M. Kuzenko, S. Theisen, JHEP **0003** (2000) 034, hep-th/0001068
- [19] M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory" (Cambridge University Press, 1987)
- [20] M. Shmakova, "One-loop corrections to the D3 brane action,"
hep-th/9906239
- [21] A. Hashimoto, I.R. Klebanov, Phys. Lett. **B381** (1996) 437, hep-th/9604065
- [22] J.H. Schwarz, Phys. Rep. **89** (1982) 223
- [23] O.D. Andreev, A.A. Tseytlin, Nucl. Phys. **B311** (1988) 205
- [24] M.R. Garousi, R.C. Myers, Nucl. Phys. **542**(1999) 73
- [25] S.V. Ketov, "N=1 and N=2 supersymmetric non abelian Born-Infeld actions from superspace", hep-th/0005265